Expressing Maxwell's equations independently of the unit systems

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Abstract. A procedure to teach Electrodynamics independently of unit systems is presented and compared with some of those given in physics literature.

1. Introduction

As it is remarked in [1], "it is a well-known fact that one major hurdle for students in a class of electromagnetism is to get familiar with the adopted unit system, and to move from one unit system to another (e.g. SI to Gaussian)". As a student and, later, as an Electrodynamics professor, I have felt myself this hurdle. However, inspired by the Jackson's book [2], I have adopted a procedure to teach electrodynamics independently of unit systems since 1985 and used it consistently in my lectures [3]. Here is a summary, including some comments regarding results from [1, 2] and some inconsistencies from [3].

2. Writing the Maxwell's equations

Before writing the Maxwell's equations we have to define the physical system itself, namely the electromagnetic field (EMF), by the interactions with other known systems. The Lorentz force may be experienced for introducing the electric charge q and the fundamental variables $\boldsymbol{E}$ and $\boldsymbol{B}$ of the EMF. The electric field $\boldsymbol{E}$ is defined by the force acting on a charge at rest in the inertial system of the laboratory. The relation between force, charge and electric field, as is considered in all known unit systems [2] is

$$\boldsymbol{F} = q\boldsymbol{E},$$

such that the introduction of an arbitrary proportional constant in this relation is not of practical interest. Different unit systems are introduced when expressing the Lorentz force acting on a moving charge q ‡

$$\boldsymbol{F} = q\boldsymbol{E} + \frac{q}{\alpha_0}\boldsymbol{v} \times \boldsymbol{B}, \tag{1}$$

‡ In [3] the magnetic field is defined within the old formalism of the *magnetic shells* or *sheets* and, the constant α_0 is introduced by the equation of the torque on an elementary shell $\boldsymbol{C} = (1/\alpha_0)I\boldsymbol{S} \times \boldsymbol{B} = \boldsymbol{m} \times \boldsymbol{B}$ where the magnetic moment is $\boldsymbol{m} = (1/\alpha_0)I\boldsymbol{S}$. From this torque one may deduce the Laplace's

Here, the constant α_0 depends on the unit system used.

The law of electric flux introduces a second constant k_e:

$$\oint_{\partial\mathcal{D}} \boldsymbol{E} \cdot \boldsymbol{dS} = k_e Q(\mathcal{D})$$

or in the local form

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = k_e \rho.$$

Ampère's law introduces a third constant k_m

$$\oint_{\Gamma} \boldsymbol{B} \cdot \boldsymbol{dl} = k_m I(\Sigma_\Gamma) \quad \text{or} \quad \boldsymbol{\nabla} \times \boldsymbol{B} = k_m \boldsymbol{j}.$$

Maxwell's equation that generalizes the Ampère's law is

$$\boldsymbol{\nabla} \times \boldsymbol{B} = k_m \left(\boldsymbol{j} + \frac{1}{k_e} \frac{\partial \boldsymbol{E}}{\partial t} \right).$$

Let's adopt, for practical reasons, the following new notations:

$$k_e = \frac{1}{\tilde{\varepsilon}_0}, \quad k_m = \frac{\tilde{\mu}_0}{\alpha_0}.$$

Here, the constants $\tilde{\varepsilon}_0$ and $\tilde{\mu}_0$ are proportional constants, their values depending on the units used. § The purpose of such definitions is to write the laws of EM with notations specific to the international unit system (SI).

The law of electromagnetic induction is written as

$$\oint_{\Gamma} \boldsymbol{E} \cdot \boldsymbol{dl} = -\frac{1}{\alpha_i} \int_{\Sigma_\Gamma} \frac{\partial \boldsymbol{B}}{\partial t} \cdot \boldsymbol{dS} \text{ or } \boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{\alpha_i} \frac{\partial \boldsymbol{B}}{\partial t}, \quad \alpha_i > 0$$

where α_i is the last constant introduced. Hence, the Maxwell's equations are

$$\frac{1}{\tilde{\mu}_0} \boldsymbol{\nabla} \times \boldsymbol{B} = \frac{1}{\alpha_0} \boldsymbol{j} + \frac{\tilde{\varepsilon}_0}{\alpha_0} \frac{\partial \boldsymbol{E}}{\partial t}, \tag{2}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{\alpha_i} \frac{\partial \boldsymbol{B}}{\partial t}, \tag{3}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{4}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{1}{\tilde{\varepsilon}_0} \rho. \tag{5}$$

force (up to a gradient) and from this force, the presence of the factor $1/\alpha_0$ in the Lorentz force expression.

§ In [3] the constant $\tilde{\mu}_0$ is introduced defining the magnetic scalar potential of an elementary magnetic shell

$$\Psi(\boldsymbol{r}) = \frac{\tilde{\mu}_0}{4\pi} \frac{\boldsymbol{m} \cdot \boldsymbol{r}}{r^3}.$$

Consequently, the factor $\tilde{\mu}_0$ occurs in the Ampére's law.

Several arguments justify the equality between α_0 and α_i. One may argue that

$$\alpha_i = \alpha_0$$

as in Jackson's book [2], but we may use directly the equation (1) considering a closed conductor Γ moving with a constant velocity $\boldsymbol{u}$ in an external nonuniform magnetic field $\boldsymbol{B}(\boldsymbol{r})$ in the laboratory system L. The electromotive force corresponding to the electric current due to the Lorentz force is [4]

$$\mathcal{E} = \frac{1}{\alpha_0}\oint_\Gamma (\boldsymbol{u}\times\boldsymbol{B})\cdot \boldsymbol{dl} = \frac{1}{\alpha_0}\frac{1}{\delta t}\oint_\Gamma (\boldsymbol{dl}\times\boldsymbol{\delta a})\cdot\boldsymbol{B} = -\frac{1}{\alpha_0}\frac{\delta N_m}{\delta t}$$

where $\boldsymbol{\delta a}$ is the displacement vector of the element $\boldsymbol{dl}$ of the current contour during the time δt, and δN_m is the variation of the magnetic flux N_m through a surface attached to this contour due to the displacement $\boldsymbol{\delta a}$. After comparing the last equation to the equation (3), and after considering that only the magnetic flux variation is the defining element of the electromagnetic induction, we have to admit the equality between the two constants α_i and α_0.

In [2] this equality is argued theoretically using the Galilei invariance of Maxwell's equations for $u << c$: for the observer in the system L' of the conductor the effect is associated with an induced electric field

$$\boldsymbol{E}' = \frac{1}{\alpha_0}\boldsymbol{u}\times\boldsymbol{B}, \quad (\boldsymbol{E}=0).$$

This equation, together with the transformation law $\boldsymbol{B}' = \boldsymbol{B}$, is, indeed, the first approximation of the relativistic transformation law.

Another argument for considering the two constants equal is given by the physical requirements of the EM theory. The definitions of charge, energy *etc* must complete the Maxwell's equations together with the corresponding theorems resulting from these equations. Combining the equations (2) and (3) one obtains

$$\frac{1}{\tilde{\mu}_0}\boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) + \tilde{\varepsilon}_0\boldsymbol{E}\times(\boldsymbol{\nabla}\times\boldsymbol{E}) = -\frac{1}{\alpha_0}\boldsymbol{j}\times\boldsymbol{B} - \frac{\tilde{\varepsilon}_0}{\alpha_0}\frac{\partial\boldsymbol{E}}{\partial t}\times\boldsymbol{B} - \frac{\tilde{\varepsilon}_0}{\alpha_i}\boldsymbol{E}\times\frac{\partial\boldsymbol{B}}{\partial t}$$

and, finally,

$$\tilde{\varepsilon}_0\left[\frac{1}{\alpha_0}\frac{\partial\boldsymbol{E}}{\partial t}\times\boldsymbol{B} + \frac{1}{\alpha_i}\boldsymbol{E}\times\frac{\partial\boldsymbol{B}}{\partial t}\right] = -\boldsymbol{\nabla}\cdot\mathsf{T} - \boldsymbol{f} \qquad (6)$$

where $\boldsymbol{f} = \rho\boldsymbol{E} + (1/\alpha_0)\boldsymbol{j}\times\boldsymbol{B}$ is the Lorentz force density, and

$$\mathsf{T}_{ik} = \frac{1}{2}\left(\tilde{\varepsilon}_0 E^2 + \frac{1}{\tilde{\mu}_0}B^2\right)\delta_{ik} - \left(\tilde{\varepsilon}_0 E_i E_k + \frac{1}{\tilde{\mu}_0}B_i B_k\right).$$

The equation (6) represents the relation between the electromagnetic forces and the Maxwell's stress tensor $\mathcal{T}_{ik} = -\mathsf{T}_{ik}$ within the static case. It can be considered as the

EM momentum theorem if the left hand term is a time derivative. Consequently, this requires $\alpha_i = \alpha_0$ and defines the electromagnetic momentum density by

$$\boldsymbol{g}_{em} = \frac{\tilde{\varepsilon}_0}{\alpha_0} \left(\boldsymbol{E} \times \boldsymbol{B} \right).$$

Let's consider $\alpha_i = \alpha_0$ from now on. We point out that the equations are written using SI units by substituting $\alpha_0 = 1$, $\tilde{\varepsilon}_0 = \varepsilon_0$, $\tilde{\mu}_0 = \mu_0$. ‖

In the case of the free EMF ($\rho = 0$, $\boldsymbol{j} = 0$) one obtains the propagation equations for $\boldsymbol{E}$ and $\boldsymbol{B}$,

$$\Delta \boldsymbol{E} - \frac{\tilde{\varepsilon}_0 \tilde{\mu}_0}{\alpha_0^2} \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0, \quad \Delta \boldsymbol{B} - \frac{\tilde{\varepsilon}_0 \tilde{\mu}_0}{\alpha_0^2} \frac{\partial^2 \boldsymbol{B}}{\partial t^2} = 0$$

From the last equations one obtains the fundamental relation

$$\frac{\alpha_0^2}{\tilde{\varepsilon}_0 \tilde{\mu}_0} = c^2, \tag{7}$$

between the three constants $\tilde{\varepsilon}_0$, $\tilde{\mu}_0$, α_0 introduced in the Maxwell's equations and an experimental constant, the light speed c. The two remaining arbitrary constants define different unit systems.

The electromagnetic potentials are introduced by the equations

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}, \quad \boldsymbol{E} = \boldsymbol{\nabla} \Phi - \frac{1}{\alpha_0} \frac{\partial \boldsymbol{A}}{\partial t}$$

and verify

$$\Delta \boldsymbol{A} - \frac{\tilde{\varepsilon}_0 \tilde{\mu}_0}{\alpha_0^2} \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\frac{\tilde{\mu}_0}{\alpha_0} \boldsymbol{j} + \boldsymbol{\nabla}. \left(\boldsymbol{\nabla} \cdot \boldsymbol{A} + \frac{\tilde{\varepsilon}_0 \tilde{\mu}_0}{\alpha_0} \frac{\partial \Phi}{\partial t} \right),$$

$$\Delta \Phi = -\frac{1}{\tilde{\varepsilon}_0} \rho - \frac{1}{\alpha_0} \frac{\partial}{\partial t} \boldsymbol{\nabla} \cdot \boldsymbol{A}.$$

The gauge transformations are

$$\boldsymbol{A} \longrightarrow \boldsymbol{A} + \boldsymbol{\nabla} \Psi, \quad \Phi \longrightarrow \Phi - \frac{1}{\alpha_0} \frac{\partial \Psi}{\partial t},$$

and the Lorenz constraint is

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} + \frac{\tilde{\varepsilon}_0 \tilde{\mu}_0}{\alpha_0} \frac{\partial \Phi}{\partial t} = 0.$$

Correspondingly, the equations of the potentials in this gauge are

$$\Delta \boldsymbol{A} - \frac{1}{c^2} \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\frac{\tilde{\mu}_0}{\alpha_0} \boldsymbol{j}, \quad \Delta \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -\frac{1}{\tilde{\varepsilon}_0} \rho$$

with the retarded solutions

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\tilde{\mu}_0}{4\pi\alpha_0} \int \frac{\boldsymbol{j}(\boldsymbol{r}', t - R/c)}{R} \mathrm{d}^3 x', \quad \Phi(\boldsymbol{r}, t) = \frac{1}{4\pi\tilde{\varepsilon}_0} \int \frac{\rho(\boldsymbol{r}', t - R/c)}{R} \mathrm{d}^3 x'. \tag{8}$$

‖ Those of my students who do not agree with my general notation are free to use this choice and, so, to work in SI units. They are notified on this freedom from the first class.

The above notations are specific for *rationalized* unit systems. To find out the changes required to rewrite the Maxwell's equations using *non rationalized* unit systems, let's consider the solutions (8) for the scalar and vector potentials. We convert the Maxwell's equations (2)-(5) to *non rationalized* notations by eliminating the factor $1/4\pi$ from (8)

$$\tilde{\varepsilon}_0 \longrightarrow \frac{\tilde{\varepsilon}_0}{4\pi}, \quad \tilde{\mu}_0 \longrightarrow 4\pi\tilde{\mu}_0.$$

The equation (7) is invariant to these transformations. The Maxwell's equations with the notations of a non rationalized unit system are

$$\frac{1}{\tilde{\mu}_0}\boldsymbol{\nabla}\times\boldsymbol{B} = \frac{4\pi}{\alpha_0}\boldsymbol{j} + \frac{\tilde{\varepsilon}_0}{\alpha_0}\frac{\partial \boldsymbol{E}}{\partial t},$$

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{1}{\alpha_0}\frac{\partial \boldsymbol{B}}{\partial t},$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0,$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \frac{4\pi}{\tilde{\varepsilon}_0}\rho.$$

The Maxwell's equations (2)-(5) with $\alpha_i = \alpha_0$ and all their consequences are rewritten in all usual unit systems by substituting the following values for the three constants:

SI:	$\alpha_0 = 1, \ \tilde{\varepsilon}_0 = \varepsilon_0 \quad , \tilde{\mu}_0 = \mu_0 \quad ,$
Heaviside:	$\alpha_0 = c, \ \tilde{\varepsilon}_0 = \tilde{\mu}_0 = 1,$
Gauss:	$\alpha_0 = c, \ \tilde{\varepsilon}_0 = \frac{1}{4\pi}, \quad \tilde{\mu}_0 = 4\pi,$
esu:	$\alpha_0 = 1, \ \tilde{\varepsilon}_0 = \frac{1}{4\pi}, \quad \tilde{\mu}_0 = \frac{4\pi}{c^2},$
emu:	$\alpha_0 = 1, \ \tilde{\varepsilon}_0 = \frac{1}{4\pi c^2}, \ \tilde{\mu}_0 = 1.$

These notations work also within the relativistic electrodynamics (in vacuum). The relativistic equations of motion of a charged particle are obtained from the Lagrange function

$$L(t) = -m_0c^2\sqrt{1-\frac{v^2}{c^2}} - q\Phi + \frac{1}{\alpha_0}q\boldsymbol{v}\cdot\boldsymbol{A}$$

which may be written with an invariant parameterization λ as

$$L(\lambda) = -m_0c\sqrt{\dot{x}_\mu(\lambda)\dot{x}^\mu(\lambda)} - \frac{1}{\alpha_0}A_\mu(x)\dot{x}^\mu$$

where

$$(A^\mu) = \left(\frac{\alpha_0}{c}\Phi, \ \boldsymbol{A}\right).$$

is the *4-potential*. The relativistic invariance of the motion equations and of the Maxwell's equations is realized by defining $T_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ as components of a tensor. In particular, A^μ may be considered as the components of a 4-vector although this is not a necessary condition of the invariance of the theory, except the request to have covariant equations for the 4-potential. ¶

As it is pointed out in [2, 1], many difficulties are encountered trying to generalize this procedure to the macroscopic electromagnetic field in the presence of a medium. The complications arise due to some inconsistencies in the definitions adopted in various unit systems as it is pointed out in [1].
In [3] one defines the vectors $\boldsymbol{D}$ and $\boldsymbol{H}$ by the equations

$$\boldsymbol{D} = \tilde{\varepsilon}_0 \boldsymbol{E} + \boldsymbol{P}, \quad \boldsymbol{H} = \frac{1}{\tilde{\mu}_0}\boldsymbol{B} - \boldsymbol{M}$$

such that one may obtain the macroscopic Maxwell's equations only in SI and Heaviside unit systems. In [1], the definitions of $\boldsymbol{D}$ and $\boldsymbol{H}$ are given by introducing two new constants α_d and α_h (in [1] labeled k_D and k_H). With our notations, they are

$$\boldsymbol{D} = \alpha_d \left(\boldsymbol{E} + \frac{1}{\tilde{\varepsilon}_0} \boldsymbol{P} \right),$$

$$\boldsymbol{H} = \alpha_h \left(\boldsymbol{B} - \tilde{\mu}_0 \boldsymbol{M} \right).$$

However, in [1] a new constant α_m (in [1] labeled k_M) is introduced by the relation between the magnetization current $\boldsymbol{j}_m$ and the magnetization vector $\boldsymbol{M}$

$$\boldsymbol{j}_m = \alpha_m \boldsymbol{\nabla} \times \boldsymbol{M}.$$

This new constant is not necessary. Actually, we can reduce the number of supplementary constants to two, as in [2].

The constant α_m is necessary in the definition of the magnetic dipolar moment (and in all multipolar orders). If we consider the Laplace's force, the corresponding expression is well defined by the Lorentz force. Furthemore, for a steady current loop Γ we have

$$\boldsymbol{F}(I, \Gamma) = \frac{I}{\alpha_0} \oint_{\Gamma} \boldsymbol{dl} \times \boldsymbol{B}$$

We may demonstrate the relation

$$\oint_{\Gamma} \boldsymbol{dl} \times \boldsymbol{B} = \int_{\Sigma(\Gamma)} (\boldsymbol{n} \cdot \boldsymbol{\nabla}) \boldsymbol{B} \mathrm{d}S.$$

So, the Laplace's force is equivalent, at least regarding the resulting force, to a fictitious force acting on the shell $\Sigma(\Gamma)$

$$\boldsymbol{dF}' = \frac{I}{\alpha_0} \boldsymbol{dS}.$$

The simplest and natural definition of the magnetic moment $\boldsymbol{dm}$ corresponding to an elementary shell is

$$\boldsymbol{dm} = \frac{I}{\alpha_0} \boldsymbol{dS}.$$

¶ Curiously, in a considerable part of the physics literature, the vectorial character of the 4-potential is presented as a necessary condition for the relativistic invariance of the theory.

Also, it is possible to relate the torque $\boldsymbol{dm}\times\boldsymbol{B}$ to the Laplace's force.[3]. Therefore, in the case of a current distribution $\boldsymbol{j}$ in $\mathcal{D}$ the magnetic dipolar moment is defined by

$$\boldsymbol{m} = \frac{1}{2\alpha_0}\int\limits_{\mathcal{D}} \boldsymbol{r}\times\boldsymbol{j}\mathrm{d}^3x.$$

Generally, one may define the $n-$th order magnetic moment by the tensor [5]

$$\mathsf{M}^{(n)} = \frac{n}{(n+1)\alpha_0}\int\limits_{\mathcal{D}} \boldsymbol{r}^n\times\boldsymbol{j}\,\mathrm{d}^3x. \tag{9}$$

The magnetization current $\boldsymbol{j}_m$ is given by the relation

$$\boldsymbol{j}_m = \alpha_0\,\boldsymbol{\nabla}\times\boldsymbol{M}$$

where $\boldsymbol{M}$ includes the contributions of all magnetic multipoles. This is a result of the definition (9) and of the average of microscopic equations of EMF. In conclusion, the equality

$$\alpha_m = \alpha_0$$

is justified while a third supplementary constant is not necessary in the case of the macroscopic field.

3. Conclusion

By writing the macroscopic Maxwell's equations as in [3],

$$\begin{aligned}
\boldsymbol{\nabla}\times\boldsymbol{H} &= \frac{1}{\alpha_0}\boldsymbol{j} + \frac{1}{\alpha_0}\frac{\partial\boldsymbol{D}}{\partial t},\\
\boldsymbol{\nabla}\times\boldsymbol{E} &= -\frac{1}{\alpha_0}\frac{\partial\boldsymbol{B}}{\partial t},\\
\boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0,\\
\boldsymbol{\nabla}\cdot\boldsymbol{D} &= \rho
\end{aligned}$$

only the equations in SI and Heaviside's systems are obtained. To change the unit system from Heaviside's to the Gaussian one, we have to memorize some factors 4π.

With notations from the present paper we have

$$\begin{aligned}
\boldsymbol{\nabla}\times\boldsymbol{H} &= \frac{\tilde{\mu}_0\alpha_h}{\alpha_0}\boldsymbol{j} + \frac{\alpha_0\alpha_h}{\alpha_d}\frac{1}{c^2}\frac{\partial\boldsymbol{D}}{\partial t},\\
\boldsymbol{\nabla}\times\boldsymbol{E} &= -\frac{1}{\alpha_0}\frac{\partial\boldsymbol{B}}{\partial t},\\
\boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0,\\
\boldsymbol{\nabla}\cdot\boldsymbol{D} &= \frac{\alpha_d}{\tilde{\varepsilon}_0}\rho,
\end{aligned} \tag{10}$$

and

$$\boldsymbol{D} = \alpha_d\left(\boldsymbol{E} + \frac{1}{\tilde{\varepsilon}_0}\boldsymbol{P}\right),\quad \boldsymbol{H} = \alpha_h\left(\boldsymbol{B} - \tilde{\mu}_0\boldsymbol{M}\right). \tag{11}$$

The equations (10) result from the microscopic equations using the relations

$$< \rho_{micro} >= -\boldsymbol{\nabla} \cdot \boldsymbol{P}, \quad < \boldsymbol{j}_{micro} >= \boldsymbol{j} + \alpha_0 \boldsymbol{\nabla} \times \boldsymbol{M} + \frac{\partial \boldsymbol{P}}{\partial t},$$

and the definitions (11). The following values for the two supplementary constants, named in [1] *conventional* constants, should be substituted to obtain the equations within different unit systems. For the two supplementary constants in the usual unit we have the following expressions:

$$\begin{array}{lll} \text{SI:} & \alpha_d = \varepsilon_0, & \alpha_h = \dfrac{1}{\mu_0}, \\ \text{Heaviside :} & \alpha_d = 1, & \alpha_h = 1, \\ \text{Gaussian :} & \alpha_d = 1, & \alpha_h = 1, \\ \text{esu :} & \alpha_d = 1, & \alpha_h = c^2, \\ \text{emu :} & \alpha_d = \dfrac{1}{c^2}, & \alpha_h = 1 \end{array}$$

Although the number of the *conventional* constants is reduced to two, the conclusion from [1] remains valid: the complications due to these *conventional* constants make the result "not as appealing as that obtained in the vacuum case...".

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